Option Price Calculator: European, American, Bermudan (binomial tree)

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Contents

1 Methodology

1.1 Price tree

The price of an underlying stock can be simulated using the binomial tree algorithm. Time axis is presented with discreet time points $t_j = j \cdot \Delta t$ (Δt is the time step and $T = n \cdot \Delta t$ is the option maturity, $j = 0 \cdots n$). Prices at the time point t_{j+1} are

$$S_{j,i} = S_0 \cdot u^{2i-j} \tag{1}$$
$$j = 0 \cdots n \tag{2}$$
$$i = 0 \cdots j \tag{3}$$

The scale factors for price moving up by u, or moving down by u^{-1} are:

$$u = e^{\sigma\sqrt{\Delta t}} \tag{4}$$
$$\sigma\text{- equity volatility} \tag{5}$$

Probabilities for price movement up (p_u) or down (p_d) are:

$$p_u = \frac{e^{(r-q)\Delta t} - u^{-1}}{u - u^{-1}} \tag{6}$$
$$p_d = 1 - p_u \tag{7}$$
$$\tag{8}$$

where r and q are interest rate and dividend rate respectively.

1.2 Migration probability tree

For each scenario the path starts at the tree root $(0, 0)$ and goes through the tree nodes switching either up ($i \to (i + 1)$) or down ($i \to i$) according to probabilities p_u and p_d. A probability P_{ji} to reach a node (j, i) is calculated as follows:

$$P_{ji} = \begin{cases} p_u^j, & i = j; \quad j = 0 \cdots, n \\ p_d^j, & i = 0; \quad j = 0 \cdots, n \\ P_{j-1,i} p_d + P_{j-1,i-1} p_u, & i = 1 \cdots, j-1; \quad j = 2 \cdots, n \end{cases} \tag{9}$$

1.3 Backward induction of option prices

European Option European option is exercised at maturity:

$$V_{ni} = \max\left[0, \pm(S_{ni} - K)\right] \tag{10}$$

Here signs $\pm$ correspond to call/put options.
The connection between the derivative price $V_{j-1,i}$ at time t_{j-1} (the node i) and derivative prices $V_{j,i+1}$, and $V_{j,i}$ at time t_j (nodes $i+1$ and i) is:

$$V_{ji} = e^{-r\Delta t}\left(p_d V_{j+1,i} + p_u V_{j+1,i+1}\right) \tag{11}$$

The equation (**??**) provides the tool for backward derivative price calculation. Derivative values at maturity (V_{ni}) are assumed to be known through payoff function. Working backward from $j = n$ to $j = 0$ with equation (**??**) one can obtain derivative values at vall tree nodes and the present value V_{00} at $t = 0$.

American Option In case of American Option the exercise can occure at any time. To take this into account we modify the node values (Equation **??**) as follows

$$V_{ji}^A = \max\left[V_{ji}, \pm(S_{ji} - K)\right] \tag{12}$$

Bermudan Option Bermudan option can be exercised in predefined time points t_k (including maturity). The node values in this case are modified at exercise time points t_k as follows:

$$V_{j_k i}^B = \max\left[V_{j_k i}, \pm(S_{j_k i} - K)\right] \tag{13}$$

where $t_k = j_k \cdot dt$.
This numerical procedure provides with option values at all tree nodes including $t = 0$:

$$V_{00} \tag{14}$$

1.4 Potential Future Exposure

The node values V_{ji} obtained in the previous section **??** and the tree probability map (section **??**) can be used for calculation of the Potential Future Exposure (PFE). The node values V_{ji} at time t_j are associated with probabilities P_{ji} of reaching nodes (j, i). An array of pairs $[V_{ji}, P_{ji}]$ is sorted by V_{ji} (in ascending order)

$$[V_{ji}, P_{ji}] \quad \xrightarrow[sorting]{} \quad [\hat{V}_{ji}, \hat{P}_{ji}] \tag{15}$$

Cumulative probability can be calculated now as

$$C_i = \sum_{k=1}^{i} P_{jk} \tag{16}$$

Given percentile η the corresponding PFE is calculated as follows

$$\begin{aligned} \hat{i} &: C_{\hat{i}} \leq \eta < C_{\hat{i}+1} \\ V_j^{\eta} &= V_{j,\hat{i}} + \frac{\eta - P_{j,\hat{i}}}{P_{j,\hat{i}+1} - P_{j,\hat{i}}}(V_{j,\hat{i}+1} - V_{j,\hat{i}}) \\ PFE_j &= V_j^{\eta} e^{-r\cdot t_j} \end{aligned} \tag{17}$$

1.5 Expected Exposure

The Expected Exposure is simply the expected value of V_{ji}:

$$EE_j = \sum_{i=0}^{2j} V_{ji} P_{ji} \cdot e^{-r\cdot t_j} \tag{18}$$

2 Examples

As an example we calculate option prices with the following parameters:

Table 1: Parameters

Parameters	Maturity	Time step	t_k	S	K	r	σ	q
Values	5y	3d	$1y, 2y, 3y, 4y$	100	100	0.05	$0.05 \cdots 0.80$	$0.02 \cdots 0.10$

Results:

Table 2: Option prices (low dividend rate $q = 0.02$)

	$\sigma = 0.05$		$\sigma = 0.30$		$\sigma = 0.80$	
Option Style	Call	Put	Call	Put	Call	Put
European	12.992	0.389	28.931	16.327	59.355	46.752
Bermudan	12.992	1.139	29.037	18.555	60.587	51.111
American	12.992	1.418	29.065	19.033	60.851	51.956

Table 3: Option prices (high dividend rate $q = 0.10$)

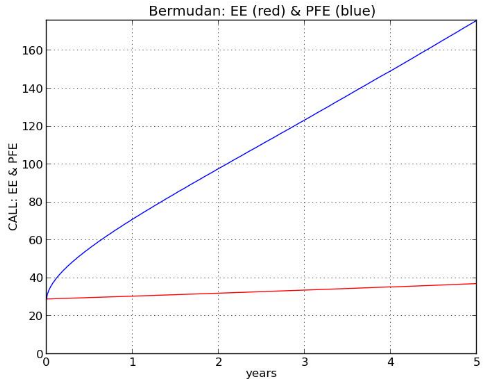
	$\sigma = 0.05$		$\sigma = 0.30$		$\sigma = 0.80$	
Option Style	Call	Put	Call	Put	Call	Put
European	0.0329	17.260	10.766	27.993	35.245	52.472
Bermudan	0.4646	17.260	14.700	28.627	43.789	56.256
American	0.8787	17.260	15.466	28.749	45.215	56.906

The time dependence of EE and of PFE is presented in Table **??** for the following set of parameters:

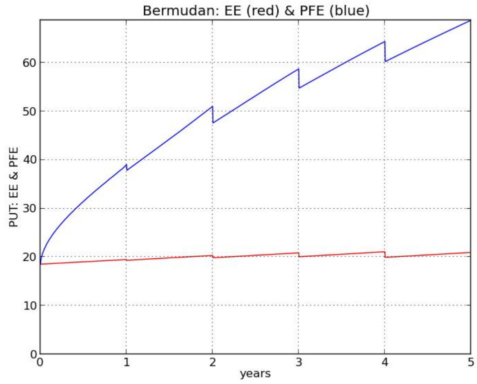
Table 4: Parameters

Parameters	Option	S	K	T	dt	σ	r	q	t_k
Values	Bermudan	100	100	5y	1d	0.30	0.05	0.02	$1y, 2y, 3y, 4y$

Table 5: Bermudan Option EE and PFE (95^{th} percentile)



Call



Put