Option Price Calculator: European, American, Bermudan (trinomial tree)

YASHKIR CONSULTING
www.yashkir.com

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1 Methodology

1.1 Price tree

The price of an underlying stock can be simulated using the trinomial tree algorithm. Time axis is presented with discreet time points $t_j = j \cdot \Delta t$ (Δt is the time step and $T = n \cdot \Delta t$ is the option maturity, $j = 0 \cdots n$). Prices at the time point t_{j+1} are

$$S_{j+1,i+2} = S_{ji} \cdot u^{+1} \tag{1}$$

$$S_{j+1,i+1} = S_{ji} \tag{2}$$

$$S_{j+1,i} = S_{ji} \cdot u^{-1} \tag{3}$$

The scale factors for price moving up by u, or moving down by u^{-1} are:

$$u = e^{\sigma\sqrt{2\Delta t}} \tag{4}$$

$$\sigma\text{- equity volatility} \tag{5}$$

Probabilities for price movement up (p_u), down (p_d) or staying the same (p_m) are:

$$p_u = \frac{\sqrt{u} \cdot e^{-(r-q)\Delta t/2} - 1}{(u-1)^2} \tag{6}$$

$$p_d = \frac{\sqrt{u} \cdot e^{-(r-q)\Delta t/2} - u}{(u-1)^2} \tag{7}$$

$$p_m = 1 - p_u - p_d \tag{8}$$

where r and q are interest rate and dividend rate respectively.
In general, the prices at time t_j at nodes i are:

$$S_{ji} = S_0 \cdot u^{i-j} \tag{9}$$
$$j = 0, \cdots, n \tag{10}$$
$$i = 0, \cdots, 2j \tag{11}$$

1.2 Migration probability tree

For each scenario the path starts at the tree root $(0,0)$ and goes through the tree nodes switching either up $(i \to (i+2))$, down $(i \to i)$ or staying the same $(i \to (i+1))$ according to probabilities p_u, p_d, and p_m. A probability P_{ji} to reach a node (j,i) is calculated as follows:

$P_{00} = 1$, $P_{11} = p_m$

$$P_{j+1,i} = \begin{cases} P_{j,i-2}p_u, & i = 2j; \quad j = 0 \cdots, n-1 \\ P_{j,i-2}p_u + P_{j,i-1}p_m, & i = 2j-1; \quad j = 2 \cdots, n-1 \\ P_{ji}p_d + P_{j,i-1}p_m + P_{j,i-2}p_u, & \text{for } i = 2 \cdots, 2j-2; \quad j = 2 \cdots, n-1 \\ P_{j,i-1}p_m + P_{j,i}p_d, & i = 1; \quad j = 2 \cdots, n \\ P_{j,i}p_d, & i = 0; \quad j = 0 \cdots, n-1 \end{cases} \tag{12}$$

1.3 Backward induction of option prices

European Option European option is exercised at maturity:

$$V_{n,i} = \max\left[0, \pm(S_{n,i} - K)\right] \tag{13}$$

Here signs $\pm$ correspond to call/put options.
The connection between a derivative price $V_{j-1,i}$ at time t_{j-1} (the node i) and derivative prices $V_{j,i+2}$, $V_{j,i+1}$, and $V_{j,i}$ at a time t_j (nodes i, $i+1$ and $i+2$) is:

$$V_{ji} = e^{-r\Delta t}\left(p_u V_{j+1,i+2} + p_m V_{j+1,i+1} + p_d V_{j+1,i}\right) \tag{14}$$

The equation (14) provides the tool for backward derivative price calculation. Derivative values at maturity ($V_{n+1,i}$) are assumed to be known through payoff function. Working backward from $j = n$ to $j = 0$ with equation (14) one can obtain the derivative price V_{00} at $t = 0$.

American Option In case of American Option the exercise can occure at any time. To take this into account we modify the node values (Equation 14) as follows

$$V_{ji}^A = \max\left[V_{ji}, \pm(S_{ji} - K)\right] \tag{15}$$

Bermudan Option A Bermudan option can be exercised in user-defined time points t_k (including maturity). The node values in this case are modified at exercise time points t_k as follows:

$$V_{j_k i}^B = \max\left[V_{j_k i}, \pm(S_{j_k i} - K)\right] \tag{16}$$

where $t_k = j_k \cdot dt$.
This numerical procedure leads to the option value at $t = 0$:

$$V_{00} \tag{17}$$

1.4 Potential Future Exposure

The node values V_{ji} obtained in the previous section 1.3 and the tree probability map (section 1.2) can be used for calculation of the Potential Future Exposure (PFE). The node values V_{ji} at time t_j are associated with probabilities P_{ji} of reaching nodes (j, i). An array of pairs $[V_{ji}, P_{ji}]$ is sorted by V_{ji} (in ascending order)

$$[V_{ji}, P_{ji}] \quad \xrightarrow[sorting]{} \quad [\hat{V}_{ji}, \hat{P}_{ji}] \tag{18}$$

Cumulative probability can be calculated now as

$$C_i = \sum_{k=1}^{i} P_{jk} \tag{19}$$

Given percentile η the corresponding PFE is calculated as follows

$$\begin{aligned} \hat{\imath} &: C_{\hat{\imath}} \le \eta < C_{\hat{\imath}+1} \\ V_j^{\eta} &= V_{j,\hat{\imath}} + \frac{\eta - P_{j,\hat{\imath}}}{P_{j,\hat{\imath}+1} - P_{j,\hat{\imath}}}(V_{j,\hat{\imath}+1} - V_{j,\hat{\imath}}) \\ PFE_j &= V_j^{\eta} e^{-r \cdot t_j} \end{aligned} \tag{20}$$

1.5 Expected Exposure

The Expected Exposure is simply the expected value of V_{ji}:

$$EE_j = \sum_{i=0}^{2j} V_{ji} P_{ji} \cdot e^{-r \cdot t_j} \tag{21}$$

2 Examples

As an example we calculate option prices with the following parameters:

Table 1: Parameters

Parameters	Maturity	Time step	t_k	S	K	r	σ	q
Values	5y	3d	$1y, 2y, 3y, 4y$	100	100	0.05	$0.05 \cdots 0.80$	$0.02 \cdots 0.10$

Results:

Table 2: Option prices (low dividend rate $q = 0.02$)

	$\sigma = 0.05$		$\sigma = 0.30$		$\sigma = 0.80$	
Option Style	Call	Put	Call	Put	Call	Put
European	12.992	0.389	28.937	16.334	59.370	46.767
Bermudan	12.992	1.137	29.044	18.561	60.600	51.125
American	12.992	1.419	29.072	19.033	60.865	51.963

Table 3: Option prices (high dividend rate $q = 0.10$)

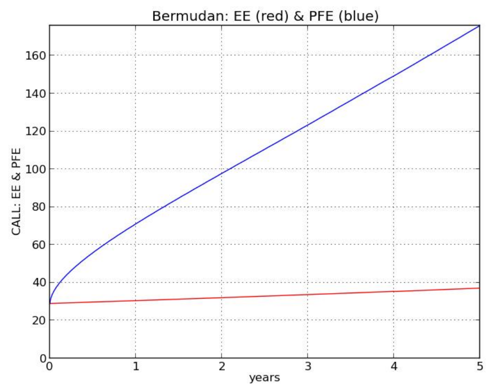
	$\sigma = 0.05$		$\sigma = 0.30$		$\sigma = 0.80$	
Option Style	Call	Put	Call	Put	Call	Put
European	0.0334	17.260	10.771	27.998	35.257	52.484
Bermudan	0.4646	17.260	14.698	28.699	43.789	56.269
American	0.8836	17.260	15.463	28.756	45.216	56.917

The time dependence of EE and of PFE is presented in Table 5 for the following set of parameters:

Table 4: Parameters

Parameters	Option	S	K	T	dt	σ	r	q	t_k
Values	Bermudan	100	100	5y	1d	0.30	0.05	0.02	$1y, 2y, 3y, 4y$

Table 5: Bermudan Option EE and PFE (95^{th} percentile)



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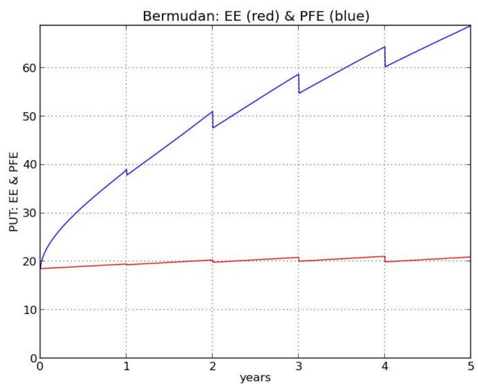

Put