Asian Option Price Calculator (modified Monte Carlo trinomial tree)

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Contents

1 Methodology

1.1 Price tree

The price of an underlying stock can be simulated using the trinomial tree algorithm. Time axis is presented with discreet time points $t_j = j \cdot \Delta t$ (Δt is the time step and $T = n \cdot \Delta t$ is the option maturity, $j = 0 \cdots n$). Prices at the time point t_{j+1} are

$$S_{j+1,i+2} = S_{ji} \cdot u^{+1} \quad (1)$$
$$S_{j+1,i+1} = S_{ji} \quad (2)$$
$$S_{j+1,i} = S_{ji} \cdot u^{-1} \quad (3)$$

The scale factors for price moving up by u, or moving down by u^{-1} are:

$$u = e^{\sigma\sqrt{2\Delta t}} \quad (4)$$
$$\sigma\text{- equity volatility} \quad (5)$$

Probabilities for price movement up (p_u), down (p_d) or staying the same (p_m) are:

$$p_u = \frac{\sqrt{u} \cdot e^{-(r-q)\Delta t/2} - 1}{(u-1)^2} \tag{6}$$

$$p_d = \frac{\sqrt{u} \cdot e^{-(r-q)\Delta t/2} - u}{(u-1)^2} \tag{7}$$

$$p_m = 1 - p_u - p_d \tag{8}$$

where r and q are interest rate and dividend rate respectively.
In general, the prices at time t_j at nodes i are:

$$S_{ji} = S_0 \cdot u^{i-j} \tag{9}$$

$$j = 0, \cdots, n \tag{10}$$

$$i = 0, \cdots, 2j \tag{11}$$

1.2 Migration probability tree

For each scenario the path starts at the tree root $(0,0)$ and goes through the tree nodes switching either up $(i \to (i+2))$, down $(i \to i)$ or staying the same $(i \to (i+1))$ according to probabilities p_u, p_d, and p_m. A probability P_{ji} to reach a node (j,i) is calculated as follows:

$P_{00} = 1$, $P_{11} = p_m$

$$P_{j+1,i} = \begin{cases} P_{j,i-2}p_u, & i = 2j; \quad j = 0 \cdots, n-1 \\ P_{j,i-2}p_u + P_{j,i-1}p_m, & i = 2j-1; \quad j = 2 \cdots, n-1 \\ P_{ji}p_d + P_{j,i-1}p_m + P_{j,i-2}p_u, & \text{for } i = 2 \cdots, 2j-2; \quad j = 2 \cdots, n-1 \\ P_{j,i-1}p_m + P_{j,i}p_d, & i = 1; \quad j = 2 \cdots, n \\ P_{j,i}p_d, & i = 0; \quad j = 0 \cdots, n-1 \end{cases} \tag{12}$$

1.3 Building the s^{th} Monte Carlo scenario

We start from the tree root node $(0,0)$. From each node (j,i) the path goes up or down according to

$$\begin{cases} \epsilon < p_d & \text{downward } (i \to i) \\ p_d < \epsilon < p_d + p_m & \text{same } (i \to i+1) \\ \epsilon > p_d + p_m & \text{upward } (i \to i+2) \end{cases} \tag{13}$$

where ϵ is a uniformly distributed random number $(0 < \epsilon < 1)$.
For each M.C. scenario s we fill the node prices sequence along the path and its average is recorded as w_{si} (here i corresponds to the end node (n,i) of the s^{th} path. As a result (after running all N Monte Carlo scenarios), we obtain prices for each end node (n,i) as average by s

$$A_i = \overline{w_{s,i}} \tag{14}$$

If number of Monte Carlo scenarios is not high enough then some of end nodes of the trinomial tree at $t = T$ remain empty (no Monte Carlo path reached those nodes). This is most likely for lowest and highest tree branches. Based on the set of average prices obtained by Monte Carlo process and on two points at $i = 0$ and $i = 2n$ (where average prices can be easily calculated because there is only one path to these end nodes) we fill the missing points of the set A_i by the spline interpolation (in practice, the best fit is obtained by using the polynomial fit of the 4^{th} degree).

1.4 Payoff for fixed/float strike Asian Option

The payoff of an Asian Option with fixed strike K is

$$V_{n,i} = \mathbf{max}\left(0, \quad \pm\left(A_i - K\right)\right) \tag{15}$$

and in case of a float strike it is

$$V_{n,i} = \mathbf{max}\left(0, \quad \pm\left(S_{n,i} - A_i\right)\right) \tag{16}$$

1.5 Backward Induction

Working back from (n, i) to $(0, 0)$

$$\begin{aligned} V_{j,i} &= \left(V_{j+1,i} \cdot p_d + V_{j+1,i+1} \cdot p_m + V_{j+1,i+2} \cdot p_u\right) e^{-r\Delta t} \\ i &= 0, \cdots, 2j \end{aligned} \tag{17}$$

we arrive to the option value at $t = 0$:

$$V_{0,0} \tag{18}$$

1.6 Potential Future Exposure

The node values V_{ji} obtained in the previous section 1.5 and the tree probability map (section 1.2) can be used for calculation of the Potential Future Exposure (PFE). The node values V_{ji} at time t_j are associated with probabilities P_{ji} of reaching nodes (j, i). An array of pairs $[V_{ji}, P_{ji}]$ is sorted by V_{ji} (in ascending order)

$$[V_{ji}, P_{ji}] \qquad \xrightarrow[sorting]{} \qquad [\hat{V}_{ji}, \hat{P}_{ji}] \tag{19}$$

Cumulative probability can be calculated now as

$$C_i = \sum_{k=1}^{i} P_{jk} \tag{20}$$

Given percentile η the corresponding PFE is calculated as follows

$$\begin{aligned} \hat{i} &: C_{\hat{i}} \leq \eta < C_{\hat{i}+1} \\ V_j^{\eta} &= V_{j,\hat{i}} + \frac{\eta - P_{j,\hat{i}}}{P_{j,\hat{i}+1} - P_{j,\hat{i}}}\left(V_{j,\hat{i}+1} - V_{j,\hat{i}}\right) \\ PFE_j &= V_j^{\eta} e^{-r \cdot t_j} \end{aligned} \tag{21}$$

1.7 Expected Exposure

The Expected Exposure is simply expected value of V_{ji}:

$$EE_j = \sum_{k=1}^{2j-1} V_{jk} P_{jk} \cdot e^{-r \cdot t_j} \tag{22}$$

2 Examples

As an example we calculate option prices with the following parameters:

Table 1: Parameters for Table 2

Parameters	Option	Notional	S	T	dt	r	q	K	Averaging	M.C. number
Values	Asian	100	100	90d	1d	0.05	0.0	98	$0 \cdots 90d$	10000

Results:

Table 2: Option prices

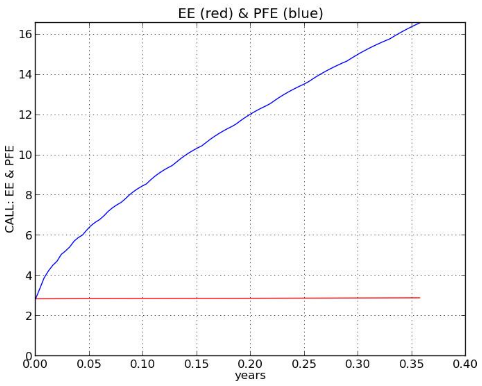
	$\sigma = 0.05$		$\sigma = 0.30$		$\sigma = 0.80$	
Strike Style	Call	Put	Call	Put	Call	Put
Fixed	2.84	0.011	5.25	2.22	8.17	4.80
Float	1.13	0.260	3.83	3.26	12.7	12.4

The time dependence of EE and of PFE is presented in Table 4 for the following set of parameters:

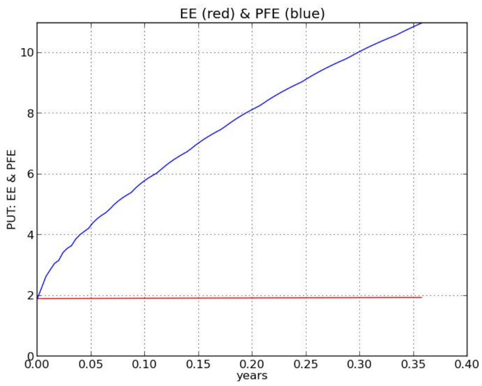
Table 3: Parameters for Table 4

Parameters	Option	Notional	S	T	dt	σ	r	q	K	K type	Averaging	M.C. number
Values	Asian	100	100	90d	1d	0.20	0.05	0.0	98	float	$0 \cdots 90d$	10000

Table 4: Asian Option EE and PFE (99^{th} percentile)



Call



Put