Barrier Option Price Calculator: European and American options (trinomial tree)

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1 Methodology

Price tree

The price of an underlying stock can be simulated using the trinomial tree algorithm. Time axis is presented with discreet time points $t_j = (j-1)\cdot\Delta t$ (Δt is the time step and $T = (n+1)\cdot\Delta t$ is the option maturity, $j = 1\cdots(n+1)$). Assuming that underlying price at t_0 is $S \equiv S_0$ prices at the next time point t_1 are $S_{2,1} = S_0\cdot u^{+1}$, $S_{2,2} = S_0$ and $S_{2,3} = S_0\cdot u^{-1}$. Following .... we find scale factors for price moving up by u, moving down by u^{-1}, or staying the same.

$$u = e^{\sigma\sqrt{2\Delta t}} \tag{1}$$

Probabilities for price movement up (p_u), down (p_d), or staying the same ($p_m = 1 - p_u - p_d$) are

$$p_u = \left(\frac{e^{\frac{1}{2}r\Delta t}\sqrt{u}-1}{u-1}\right)^2 \qquad p_d = \left(\frac{e^{\frac{1}{2}r\Delta t}\sqrt{u}-u}{u-1}\right)^2 \tag{2}$$

In general, the prices at time t_j at nodes i are:

$$S_{ji} = S_0\cdot u^{i-j} \tag{3}$$

where $j = 1,\cdots,(n+1)$; $i = 1,\cdots,(2j-1)$.

Time step adjustment Given barrier H falls somewhere between tree nodes. For increasing of the pricer precision the tree can be adjusted. We find first the node number i^* such that

$$|H - S_{n+1,i^*}| = \min_i |H - S_{n+1,i}| \tag{4}$$

From the required condition

$$H = Su^{i^*-n-1} \tag{5}$$

we find the adjusted time step

$$\Delta t = \frac{1}{2}\left(\frac{\log\frac{H}{S}}{\sigma(i^*-n-1)}\right)^2 \tag{6}$$

Modified time step should be applied again to (1), (2), and (3).

Backward calculation of option prices

The connection between a derivative price $V_{j-1,i}$ at time t_{j-1} (the node i) and derivative prices $V_{j,i+2}$, $V_{j,i+1}$, and $V_{j,i}$ at a time t_j (nodes i, $i+1$ and $i+2$) is:

$$V_{j,i} = e^{-r\Delta t}\left(p_u V_{j+1,i+2} + p_m V_{j+1,i+1} + p_d V_{j+1,i}\right) \tag{7}$$

The equation (7) provides the tool for backward derivative price calculation. Derivative values at maturity ($V_{n+1,i}$) are assumed to be known through payoff function. Working backward from $j = n$ to $j = 1$ with equation (7) one can obtain the derivative price V_{11} at $t = 0$. Account of barriers (for options of "out", "down-and-out", "up-and-out" types) is provided by setting option price V_{ji} at a node (j, i) to zero if underlying price

$$S_{ji} >= U \qquad S_{ji} <= H \tag{8}$$

and

$$t_1^H \leq t_j \leq t_2^H \qquad t_1^U \leq t_j \leq t_2^U \tag{9}$$

where the lower barrier is active between t_1^H and t_2^H and the upper barrier is active between t_1^U and t_2^U.

Payoff functions

Given set of underlying prices S_{ji} at time t_j the option value (if exercised at this time) are defined by a payoff function.
European option is exercised at maturity ($c = 1$ for a call option, $c = -1$ for a put option):

$$V_{n+1,i}^E = \max\left[0, c \cdot (S_{n+1,i} - K)\right] \tag{10}$$

An American option can be exercised at any time t_j:

$$V_{j,i}^A = \max\left[0, c \cdot (S_{j,i} - K)\right] \tag{11}$$

The conditions (8) and (9)are applied.

Option prices

We start from calculation of option prices at maturity using payoff function (10). Using backward algorithm (formula (7)) we calculate option prices as follows.
For an European option option option values are calculated using (7) from t_n to t_1.

For an American option at each node the option value is taken as maximum between (7) and (11):

$$V_{t_j,i} = \max\left[V_{ji}, V_{j,i}^A\right] \tag{12}$$

The conditions (8) and (9)are applied. The option price is equal to V_{11}.
Option values for "IN" types ("in", "up-and-in", and "down-and-in") are calculated using values of vanilla options (no barriers) and "OUT" options ("out", "up-and-out", and "down-and-out"):

$$V_{IN} = V_{vanilla} - V_{OUT} \tag{13}$$

2 Examples

As an example we calculate option prices with the following parameters:

$T = 12m$ (maturity)

$\Delta T = 1d$ (time step)

$S = 100$ (spot price)

$K = 100$ (strike price)

$r = 0.10$ (risk-free rate)

$\sigma = 0.25$ (volatility)

$q = 0.05$ (annualized dividend rate)

Results:

Table 1: Option prices; barriers are active from 0m to 12m

Barriers			European		American	
Type	H	U	Call	Put	Call	Put
out	50	140	4.0950	6.8700	11.421	7.721
out	90	110	0.0108	0.0098	5.6140	4.820
in	50	150	5.606	0.1922		
in	90	110	11.713	7.0790		
down.out	50		11.729	6.935	11.729	7.747
down.out	90		8.671	0.0774	8.671	5.702
down.in	50		0.000	0.1551		
down.in	90		3.070	7.0170		

Table 2: Option prices; barriers are active from 1m to 6m

Barriers			European		American	
Type	H	U	Call	Put	Call	Put
out	50	140	9.207	7.067	11.541	7.725
out	90	110	0.717	0.4945	5.983	5.036
in	50	150	1.188	0.0048		
in	90	110	11.007	6.594		
down.out	50		11.729	7.089	11.729	7.747
down.out	90		9.055	1.266	9.055	6.037
down.in	50		0.000	0.0017		
down.in	90		2.686	5.828		

Table 3: Option prices; barriers are active from 6m to 12m

Barriers			European		American	
Type	H	U	Call	Put	Call	Put
out	50	150	6.202	6.900	11.626	7.742
out	90	110	0.0577	0.0520	9.251	6.805
in	50	150	5.518	0.1869		
in	90	110	11.667	7.036		
down.out	50		11.729	6.938	11.729	7.747
down.out	90		10.768	0.1319	10.768	7.259
down.in	50		0.0000	0.1522		
down.in	90		0.9736	6.963		