Credit Default Swap Pricing

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1.1 Pricing model

The credit rating migration tree: probability map

The value of the Credit Default Swap can be simulated using the rating-based tree algorithm. Time axis is presented with discreet time points $t_j = i \cdot \Delta t$ (Δt is the time step and $T = n \cdot \Delta t$ is the option maturity, $i = 0 \cdots n$). Assuming that the credit rating of the underlying asset is R^* and that credit migration is governed by a risk-neutral transition matrices $\hat{\mathbf{M}}^{(i)}$ we build the probability map p_{ki} of the credit migration (probabilities to reach the node (k, i), where $k = 1 \cdots k_m$ corresponds to a credit rating R_k). We consider here the set of credit ratings as follows

$$R_k = [AAA, \quad AA, \quad A, \quad BBB, \quad BB, \quad B, \quad C, \quad D] \tag{1}$$

The migration probability is

$$\begin{aligned} p_{k,1} &= \hat{M}^{(i)}_{k_0,k} \\ p_{k,i+1} &= \sum_{j=1}^{k_m-1} p_{ji} \hat{M}^{(i+1)}_{jk} \\ k &= 1 \cdots k_m \\ i &= 1 \cdots (n-1) \end{aligned} \tag{2}$$

Here index k_0 corresponds to the initial (at t=0) credit rating R^* ($R_{k_0} \equiv R^*$) .

The credit rating migration tree: CDS values map

The underlying asset at some future time t_i may have one of R_k credit ratings. For each of these rating (nodes (k, i)) we assign CDS values V_{ki} according to the following backward induction procedure

$$
\begin{aligned}
&t = t_n = T \qquad V_{k,n} = 0, \qquad k = 1 \cdots (k_m - 1); \qquad V_{k_m,n} = P \\
&t = t_i \quad (i = n-1, \cdots 0) \qquad V_{ki} = \sum_{j=1}^{k_m} V_{j,i+1} \hat{M}_{kj}^{(i+1)} e^{-r_{i+1}\Delta t}, \qquad k = 1 \cdots (k_m - 1) \\
&V_{i,k_m} = P \equiv N \cdot (1 - \rho), \qquad i = 1 \cdots k_m
\end{aligned} \tag{3}
$$

Here N is a notional ("calculation amount"), r_i is the interest rate, and ρ is the recovery rate. The meaning of this structure is as follows. The CDS values at maturity are obviously zeroes for all non-default ratings ($k < k_m$) and the CDS value at the default rating $\mathbf{D} = R_{k_m}$ is the value of the counterparty payoff P due to the default event. The CDS value V_{ki} at time t_i at a rating k is considered as an expected CDS value at t_{i+1} discounted from t_{i+1} to t_i. The CDS values for initial ratings $k = 1 \cdots (k_m - 1)$ (or CDS premium) are found as $V_{k,1}$.

The CDS premium Fixed payments F made periodically by the buyer must provide full compensation of the CDS value at t=0. Therefore

$$
F_{CDS} = \frac{P}{\sum\limits_{i=1}^{n} e^{-r_i t_i}}. \tag{4}
$$

The fixed payments are often expressed as annualized fixed rates

$$
f_{CDS} = \left(\Delta t \sum_{i=1}^{n} e^{-r_i t_i} \right)^{-1}. \tag{5}
$$

CDS Expected and Potential Future Exposure At any future time point t_i we have k_m values V_{ki} with corresponding probabilities p_{ki}. Expected CDS exposure at a future time t_i is equal to

$$
EE_i = \sum_{k=1}^{k_m} V_{ki} p_{ki} \cdot e^{-r_i t_i} \tag{6}
$$

The Potential Future Exposure (at a given confidence level q) we calculate as follows. For a given time point t_i create a list of k_m pairs:

$$
\{V_{ki} \quad p_{ki}\} \tag{7}
$$

Sort this list by V_{ki} (so that $V_{ki} <= V_{k+1,i}$), calculate cumulative probability as

$$
C_{ki} = \sum_{j=1}^{k} p_{ji} \tag{8}
$$

Then we find the index k^* such that $C_{k^*i} \leq q \leq C_{k^*+1,i}$. The CDS PFE value is then calculated as the following interpolation:

$$
PFE_i = V_{k^*i} + \frac{q - C_{k^*i}}{C_{k^*+1,i} - C_{k^*i}} (V_{k^*+1,i} - V_{k^*i}) \tag{9}
$$

Risk-neutral transition matrices

We start from the transition matrix $T_{kk‘}$ for a given time period Δt. Transition matrices $\mathbf{M}^{(i)}$ corresponding to time periods of (t_0, t_i) are calculated as

$$\begin{aligned} \mathbf{M}^{(1)} &= \mathbf{T} \\ \mathbf{M}^{(i+1)} &= \mathbf{M}^{(i)} \times \mathbf{T} \quad i = 1, ...(n-1) \end{aligned} \tag{10}$$

Given credit spread data S_{ki} we calculate default probabilities at t_i as

$$\delta_{ki} = \frac{1 - e^{-S_{ki} \cdot \Delta t}}{1 - \rho} \tag{11}$$

The risk neutral transition matrix $R(i)$ for the time period of t_0 to t_i is then calculated as:

$$\begin{aligned} R^{(i)}_{k,k_m} &= \delta_{k,i} \\ R^{(i)}_{k,k‘} &= M^{(i)}_{k,k‘} \frac{1 - \delta_{ki}}{\sum\limits_{k”=1}^{k_m - 1} M^{(i)}_{k,k”}} \\ k = 1...(k_m - 1) &\quad k‘ = 1...(k_m - 1) \end{aligned} \tag{12}$$

Finally, we calculate marginal risk neutral transition matrices $\bar{M}^{(i)}$ which correspond to credit rating migration in the time interval (t_{i-1}, t_i):

$$\hat{M}^{(i)} = R^{(i)} \times \left(R^{(i-1)} \right)^{-1} \tag{13}$$

Inverse matrix calculation results sometimes in appearing of small negative elements. We replace those with zeroes. Default probabilities $(\hat{M}_{k,k_m})$ may also require adjustment: if $\hat{M}_{k,k_m} < \hat{M}_{k-1,k_m}$ then

$$\hat{M}_{k,k_m} = \frac{1}{2} \left(\hat{M}_{k-1,k_m} + \hat{M}_{k+1,k_m} \right) \tag{14}$$

Such a correction requires renormalization of the matrix:

$$\hat{M}^{(i)}_{k,k‘} \leftarrow \hat{M}^{(i)}_{k,k‘} \frac{1 - \hat{M}^{(i)}_{k,k_m}}{\sum\limits_{k‘=1}^{k_m - 1} \hat{M}^{(i)}_{k,k‘}} \tag{15}$$

Long/Short CDS position For the CDS buyer/seller (long/short position) the CDS values V_{ki} (3) must be replaced with

$$V_{ki} \to \mathbf{max} \left(0, \pm (V_{ki} - F_{CDS}) \right) \tag{16}$$

1.2 Input data

Rating - initial underlying credit rating

T - CDS maturity

Δt - payment frequency (time period between payments)

N - CDS notional

R - Recovery rate in case og the underlying default

η - confidence level

Buy or **Sell**

Credit Rating Transition matrix

Table 1: Example of the transition matrix, 3m period

AAA	AA	A	BBB	BB	B	C	D
0.8892	0.0926	0.0093	0.0059	0.0009	0.0019	0.0001	0.00002
0.0931	0.7663	0.1331	0.0027	0.0035	0.0008	0.0005	0.00004
0.0000	0.1097	0.7588	0.1190	0.0105	0.0029	0.0001	0.00018
0.0026	0.0000	0.1243	0.7453	0.1202	0.0076	0.0020	0.00038
0.0000	0.0033	0.0000	0.1233	0.7604	0.1136	0.0000	0.00167
0.0001	0.0000	0.0026	0.0000	0.0974	0.8205	0.0777	0.00262
0.0000	0.0001	0.0000	0.0015	0.0000	0.0673	0.9028	0.02874
0	0	0	0	0	0	0	1

Credit spreads

Table 2: Example of the spread table (the first column is time in years)

	AAA	AA	A	BBB	BB	B	C
1	0.00396	0.00522	0.00652	0.00889	0.02157	0.03181	0.04507
2	0.00446	0.00582	0.00710	0.00979	0.02386	0.03676	0.04992
3	0.00501	0.00647	0.00782	0.01071	0.02574	0.04070	0.05416
4	0.00560	0.00715	0.00865	0.01162	0.02732	0.04385	0.05790
5	0.00620	0.00784	0.00952	0.01252	0.02868	0.04647	0.06126
6	0.00682	0.00852	0.01039	0.01339	0.02994	0.04878	0.06434
7	0.00742	0.00917	0.01121	0.01421	0.03119	0.05101	0.06726
8	0.00800	0.00977	0.01193	0.01496	0.03253	0.05341	0.07013
9	0.00854	0.01031	0.01251	0.01564	0.03406	0.05620	0.07307
10	0.00901	0.01075	0.01288	0.01623	0.03588	0.05962	0.07619

Yield curve

Table 3: Example of a yield curve

tenor,y	yield
0	0.0010
1	0.0020
2	0.0027
3	0.0042
5	0.0088
7	0.0140
10	0.0204
30	0.0324

2 Examples

As an example we calculate the CDS values with the following parameters:

AAA to **C** - initial underlying credit ratings

5 - CDS maturity (years)

3m - payment frequency (time period between payments)

100 - CDS notional

0.40 - Recovery rate in case of the underlying default

0.99 - confidence level

Buy or **Sell**

Credit Rating Transition matrix as in table 1

Credit spreads as in table 2

Yield curve as in table 3

Results:

Table 4: CDS values

Rating	CDS Value	Fixed payment
AAA	0.1316	0.0068
AA	0.1834	0.0095
A	0.2714	0.0140
BBB	0.4269	0.0220
BB	0.8978	0.0463
B	1.2895	0.0665
C	1.8411	0.0950

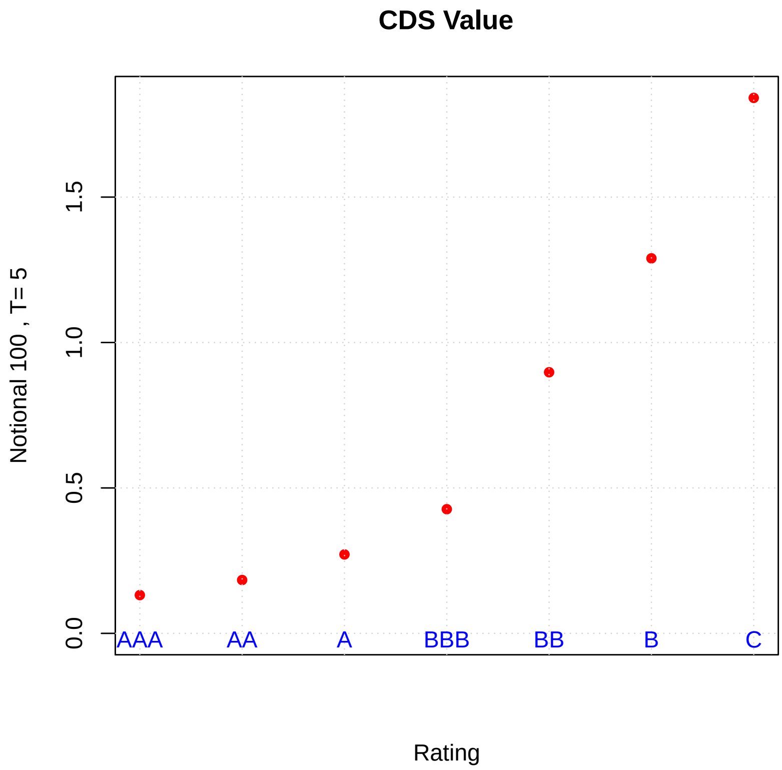

Figure 1: CDS values

The PFE for **BB** is presented in Figures 2 and 3

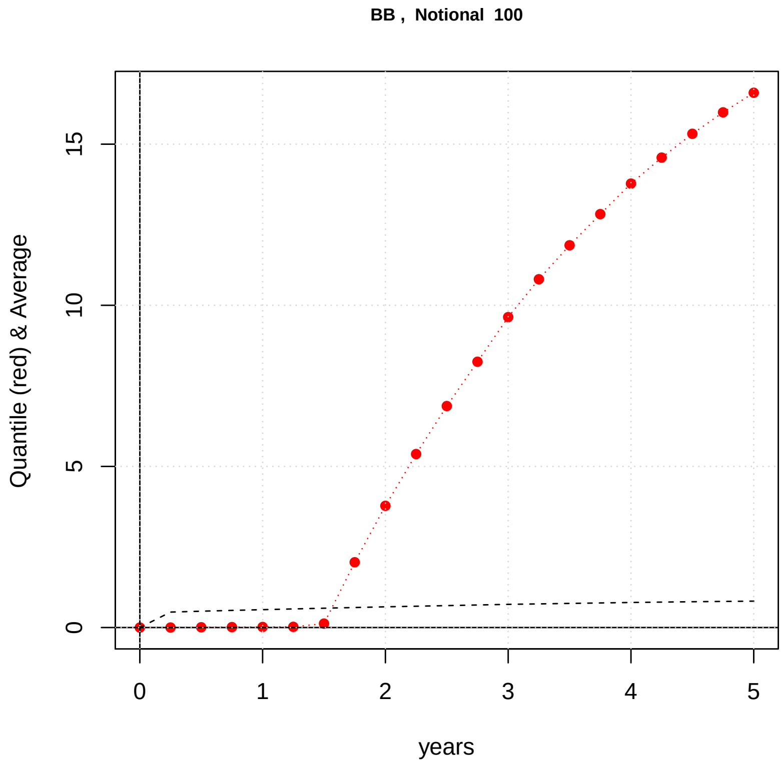

Figure 2: PFE and EE for BB at 99% confidence level. Long position. Fixed annual payments are $F_{CDS} = 0.898$

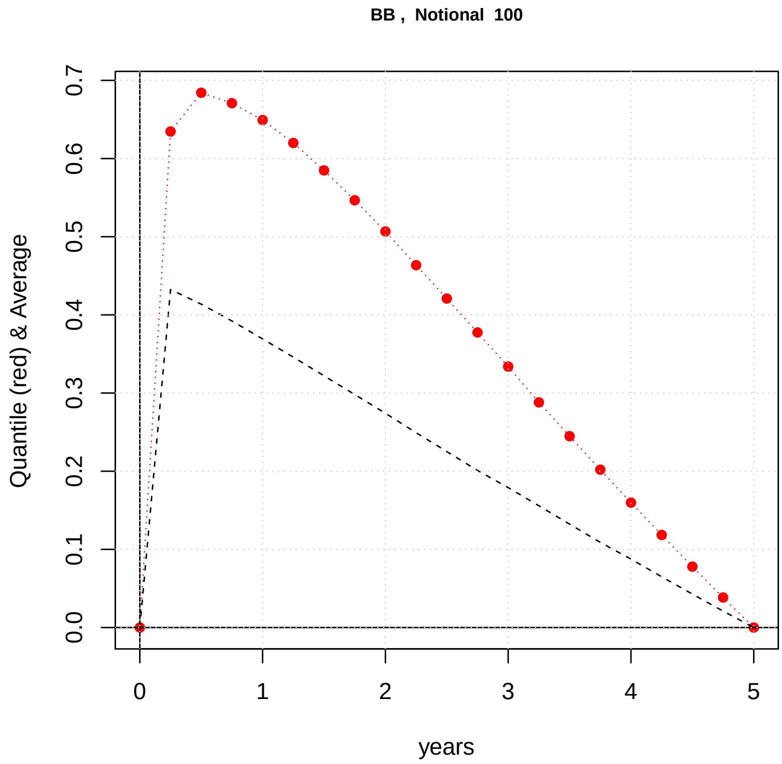

Figure 3: PFE and EE for BB at 99% confidence level. Short position. Fixed annual payments are $F_{CDS} = 0.898$

The probability map for credit rating migration see in Figure 4

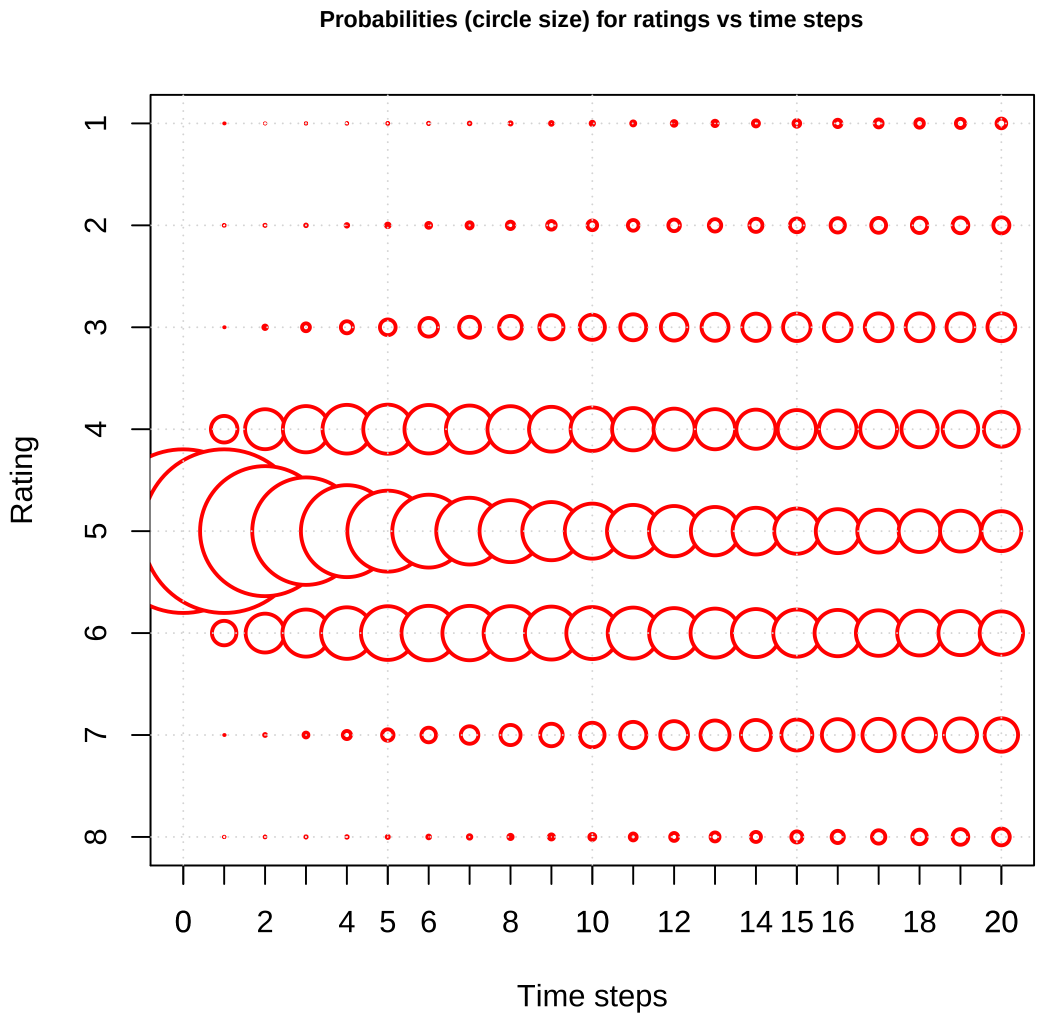

Figure 4: Probability map for initial rating BB

The time dependence of PFE for different initial credit ratings is presented in Figure 5.

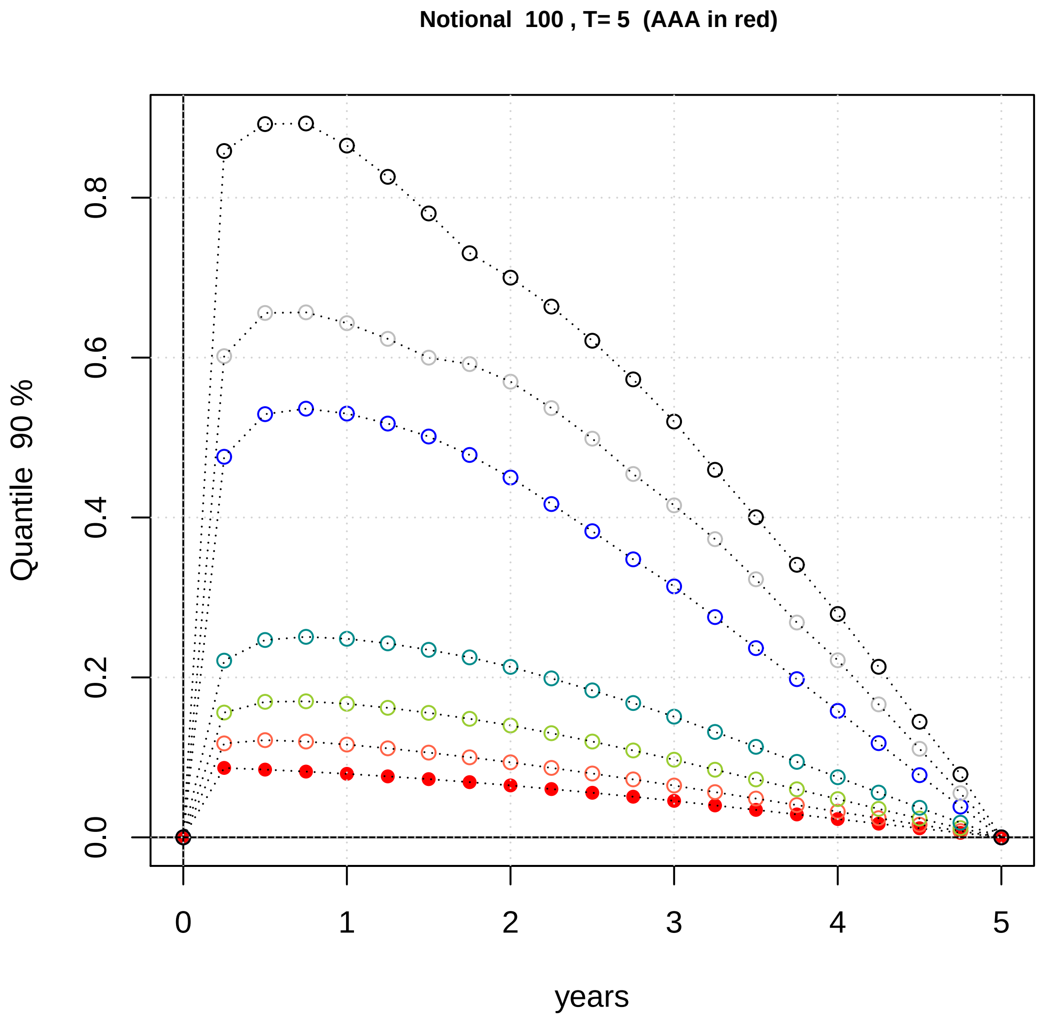

Figure 5: PFE at 90% confidence