Equity Swap Price Calculator (modified Monte Carlo trinomial tree)

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Contents

1 Introduction

An equity swap is a financial derivative contract (a swap) where a set of future cash flows are agreed to be exchanged between two counterparties at set dates in the future. The two cash flows are usually referred to as "legs" of the swap; we consider here one of these "legs" as a "fixed rate leg". The other leg of the swap is based on the performance of either a share of stock or a stock market index. This leg is commonly referred to as the "equity leg". An equity swap involves a notional principal P, a specified tenor T and predetermined payment intervals Δt. The "long" ("short") position is that receiving "equity leg" ("fixed leg") payments.

2 Methodology

$$Terminology: \begin{cases} N & \text{Notional} \\ T & \text{Maturity} \\ Type & \text{Notional: fixed or float} \\ Position & \text{long or short} \\ \Delta t & \text{Swap period} \\ k_m & \text{Number of swap periods} \\ r_0 & \text{Fixed payment rate} \\ dt & \text{Time step} \\ r & \text{Interest rate} \\ q & \text{Dividend rate} \\ \sigma & \text{Volatility} \\ M & \text{Number of Monte Carlo scenarios} \end{cases} \tag{1}$$

2.1 Price tree

The price of an underlying equity (with volatilitty of σ) can be simulated using the trinomial tree algorithm. Time axis is presented with discreet time points $t_j = j \cdot dt$ (dt is the time step and $T = n \cdot dt$ is the option maturity, $j = 0 \cdots n$). Equity prices at time points t_j and tree nodes i are

$$S_{ji} = S_0 \cdot u^{i-j} \tag{2}$$

$$j = 0 \cdots n \tag{3}$$

$$i = 0 \cdots 2j \tag{4}$$

The scale factors for price moving up by u, or moving down by u^{-1} are:

$$u = e^{\sigma\sqrt{2dt}} \tag{5}$$

Probabilities for price movement up (p_u), down (p_d) or staying the same (p_m) are:

$$p_u = \frac{\sqrt{u} \cdot e^{-(r-q)dt/2} - 1}{(u-1)^2} \tag{6}$$

$$p_d = \frac{\sqrt{u} \cdot e^{-(r-q)dt/2} - u}{(u-1)^2} \tag{7}$$

$$p_m = 1 - p_u - p_d \tag{8}$$

The Equity Swap contract (long/short position $\pm$) cash flow at a payment date $t_k = \Delta t \cdot k$ (the tree node (j, i)) for a given Monte Carlo scenario s is:

$$V_{ki}^{(s)} = \pm N_k^{(s)} \left(\frac{S_{ji}}{S_{j-1,i_{j-1}^{(s)}}} - 1 - r_0 \right) \tag{9}$$

Where:

$$N_k^{(s)} \text{ is the notional value } (N_k^{(s)} \equiv N \text{ in case of the fixed notional}). \tag{10}$$

$$S_{j-1,i_{j-1}^{(s)}} \text{ is the equity price at previous time point } t_{j-1} \text{ at a previous tree node } (i_{j-1}^{(s)}). \tag{11}$$

In case of the floating notional it is reset at time t_{j-1} to be used at t_j. Reset formula is as follows:

$$N_k = N_{k-1} \frac{S_{ji}}{S_{j-1,i_{j-1}^{(s)}}} \tag{12}$$

In the following two sections below (2.2, 2.3) the Monte Carlo process is described in details. As a result, at each payment time t_k for each tree node (i) the array of prices $V_{ki}^{(s)}$ is obtained (its length being equal to the number of Monte Carlo paths through this node). Finally, averaging by s leads to cash flow values to be used in the tree pricing procedure

$$w_{ki} = \overline{V_{ki}^{(s)}} \tag{13}$$

2.2 Migration probability tree

For each scenario the path starts at the tree root $(0,0)$ and goes through the tree nodes switching either up $(i \to (i+2))$, down $(i \to i)$ or staying the same $(i \to (i+1))$ according to probabilities p_u, p_d, and p_m. The probability P_{ji} to reach a node (j,i) is calculated as follows:

$P_{00} = 1$, $P_{11} = p_m$

$$P_{j+1,i} = \begin{cases} P_{j,i-2}p_u, & i = 2j;\ \ j = 0\cdots,n-1 \\ P_{j,i-2}p_u + P_{j,i-1}p_m, & i = 2j-1;\ \ j = 2\cdots,n-1 \\ P_{ji}p_d + P_{j,i-1}p_m + P_{j,i-2}p_u, & \text{for } i = 2\cdots,2j-2;\ \ j = 2\cdots,n-1 \\ P_{j,i-1}p_m + P_{j,i}p_d, & i = 1;\ \ j = 2\cdots,n \\ P_{j,i}p_d, & i = 0;\ \ j = 0\cdots,n-1 \end{cases} \tag{14}$$

2.3 Building the s^{th} Monte Carlo scenario

We start from the tree root node $(0,0)$. From each node (j,i) the path goes up or down according to the random value of ϵ

$$\begin{cases} \epsilon < p_d & \text{downward } (i \to i) \\ p_d < \epsilon < p_d + p_m & \text{same } (i \to i+1) \\ \epsilon > p_d + p_m & \text{upward } (i \to i+2) \end{cases} \tag{15}$$

where ϵ is a uniformly distributed random number $(0 < \epsilon < 1)$.
For each M.C. scenario s we add $V_{ki}^{(s)}$ to the sequence of prices for nodes (i) at payment times t_k along the Monte Carlo path. After all Monte Carlo scenarios are done we take average of s-sequences and record it as in (13).

2.4 Backward Induction

Working back from (n, i) to the tree root $(0, 0)$:
At maturity we have node values $Q_{n,i}$:

$$Q_{n,i} = w_{k_m,i} \tag{16}$$

Backward induction $(j = n - 1) \rightarrow (j = 0)$:

$$Q_{j,i} = (Q_{j+1,i} \cdot p_d + Q_{j+1,i+1} \cdot p_m + Q_{j+1,i+2} \cdot p_u)\, \mathbf{e}^{-r \cdot dt} \tag{17}$$

$$i = 0, \cdots, 2j \tag{18}$$

At each payment date t_k $(k = k_m \cdots k = 1)$ we add values of the cash flows:

$$Q_{j,i} \leftarrow Q_{j,i} + w_{ki} \tag{19}$$

Finally, we obtain all node values $Q_{j,i}$ including the Present Value (PV) of the Equity Swap:

$$Q_{0,0} \tag{20}$$

2.5 Potential Future Exposure of the Equity Swap

The node values Q_{ji} obtained in the previous section 2.4 and the tree probability map (section 2.2) can be used for calculation of the Potential Future Exposure (PFE). The node values Q_{ji} at time t_j are associated with probabilities P_{ji} of reaching nodes (j, i). An array of pairs $[Q_{ji}, P_{ji}]$ is sorted by Q_{ji} (in ascending order)

$$[Q_{ji}, P_{ji}] \quad \xrightarrow[sorting]{} \quad [\hat{Q}_{ji}, \hat{P}_{ji}] \tag{21}$$

Cumulative probability can be calculated now as

$$C_i = \sum_{k=1}^{i} \hat{P}_{jk} \tag{22}$$

Given percentile η the corresponding PFE is calculated as follows

$$\begin{aligned} \hat{i} &: C_{\hat{i}} \leq \eta < C_{\hat{i}+1} \\ Q_j^{\eta} &= Q_{j,\hat{i}} + \frac{\eta - P_{j,\hat{i}}}{P_{j,\hat{i}+1} - P_{j,\hat{i}}} (Q_{j,\hat{i}+1} - Q_{j,\hat{i}}) \\ PFE_j &= Q_j^{\eta} e^{-r \cdot t_j} \end{aligned} \tag{23}$$

2.6 Expected Exposure

The Expected Exposure is simply an expected value of Q_{ji}:

$$EE_j = \sum_{k=0}^{2j} Q_{jk} P_{jk} \cdot e^{-r \cdot t_j} \tag{24}$$

2.7 Swap Rate

The estimation of the Equity Swap rate r_{EQS} can be done using Expected Exposure at $t = 0$ calculated with zero fixed rate (the equity leg only) based on the assumption that the present value $EE_0^{(0)}$ of the swap (receive equity leg payments) must be offset by future fixed rate payments. The result is:

$$r_{EQS} = \frac{EE_1^{(0)}}{\sum_{k=1}^{k_m} \Delta t \cdot \mathbf{e}^{-r \cdot t_k}} \tag{25}$$

3 Examples

Table 1: General set of parameters

Position	long
Notional	1
Maturity	1y
Swap period	3m
Time step	1d
Volatility	0.25
Interest rate	0.05
Confidence level	0.99
Monte Carlo Scenarios	10000

Let first the fixed payment rate be zero ($r_0 = 0$) . Test results for present values of the Equity Swap (PV_{fx} for fixed notional and PV_{fl} for floating notional) are presented in Tables 2, 3, and 4. The Swap Rate is the fixed leg rate for which $PV = 0$.

Table 2: Fixed notional, fixed rate = 0

q	PV_{fx}	Swap Rate
0	0.0014	0.010
0.02	0.00069	0.005
0.05	-0.00026	-0.001
0.10	-0.0020	-0.015

Table 3: Floating notional, fixed rate = 0

q	PV_{fx}	Swap Rate
0	0.0009	0.008
0.02	0.00022	0.002
0.05	0.0008	-0.005
0.10	0.0023	-0.020

Table 4: Fixed/float notional, fixed rate $r_0 = 0.005$

q	PV_{fx}	PV_{fl}
0	0.0007	0.0003
0.02	0.0000	-0.0004
0.05	-0.0010	-0.0015
0.10	-0.0027	-0.0031

The PFE profile for $q = 0.02$ and $r_0 = 0.005$ (long position, fixed notional) is presented in Figure 1.

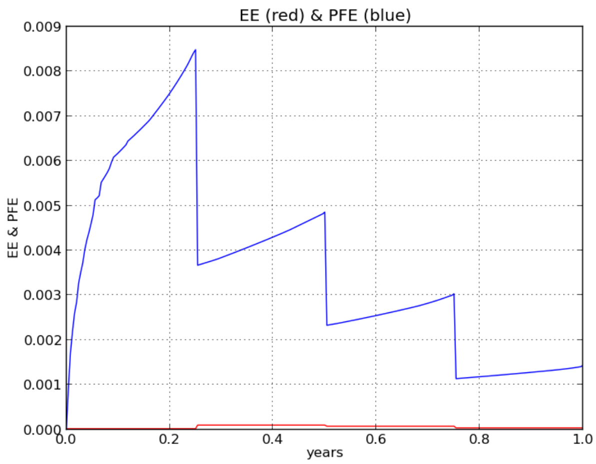

Figure 1: Equity Swap PFE