Evaluation of the power of a square matrix

To calculate $B = A^r$ (with an arbitrary real r) one can use an obvious identity formula $A = e^{\log A} \equiv e^Q$.

1. Calculate the matrix-generator Q (a logarithm of the initial matrix A) using the following Taylor series expansion for $\log A$:

$$Q = \sum_{k=1}^{n} (-1)^{k+1} \frac{D^k}{k}$$

where $D = A - I$. The upper limit n is defined by the criteria $|e^Q - A| \le \varepsilon$ where ε is the required accuracy.

2. The required power of the initial matrix is then calculated as

$$B = e^{rQ}$$

using the Taylor series

$$B = I + \sum_{k=1}^{n} \frac{(rQ)^k}{k!}$$

Example

Initial matrix **A** represents probabilities of transitions between 8 rating states (from *AAA* to *Default*) during a 1-year period.

A	*AAA*	*AA*	*A*	*BBB*	*BB*	*B*	*C*	*Default*
AAA	0.949498	0.045835	0.003621	0.001045	0.000500	0.000200	0.000100	0.000050
AA	0.004249	0.923669	0.067663	0.003524	0.000176	0.000353	0.000198	0.000167
A	0.000447	0.018313	0.933776	0.043805	0.001798	0.000774	0.000543	0.000544
BBB	0.000023	0.001789	0.039034	0.915315	0.035780	0.003859	0.002043	0.001950
BB	0.000333	0.000096	0.003561	0.068128	0.846176	0.061182	0.010828	0.008832
B	0.000001	0.000152	0.002643	0.005722	0.072033	0.823138	0.038742	0.057569
C	0.000001	0.000001	0.003333	0.002938	0.016958	0.138915	0.579544	0.258311
Default	0	0	0	0	0	0	0	1

Matrix-generator:

Q	*AAA*	*AA*	*A*	*BBB*	*BB*	*B*	*C*	*Default*
AAA	-0.051930	0.048930	0.002053	0.000973	0.000523	0.000185	0.000115	0.000023
AA	0.004522	-0.080240	0.072854	0.002094	0.000076	0.000347	0.000227	0.000110
A	0.000424	0.019684	-0.070230	0.047368	0.001017	0.000694	0.000635	0.000405
BBB	0.000226	0.001504	0.042156	-0.091040	0.040556	0.002759	0.002380	0.001450
BB	0.000359	0.000981	0.002201	0.077341	-0.171850	0.072249	0.013194	0.005525
B	-0.000006	0.000097	0.002703	0.003158	0.086144	-0.202770	0.055429	0.055249
C	0.000008	0.000065	0.004110	0.002724	0.014988	0.200370	-0.551930	0.329808
Default	0	0	0	0	0	0	0	0

Transitional matrix for one-month transitional probabilities $\mathbf{B} = \mathbf{A}^{1/2}$:

B	*AAA*	*AA*	*A*	*BBB*	*BB*	*B*	*C*	*Default*
AAA	0.995610	0.004054	0.000182	0.000082	0.000044	0.000016	0.000010	0.000002
AA	0.000374	0.993341	0.006033	0.000185	0.000007	0.000029	0.000019	0.000010
A	0.000036	0.001630	0.994176	0.003921	0.000091	0.000058	0.000052	0.000035
BBB	0.000019	0.000127	0.003490	0.992460	0.003344	0.000238	0.000195	0.000124
BB	0.000030	0.000082	0.000193	0.006376	0.985813	0.005937	0.001081	0.000486
B	0	0.000009	0.000224	0.000283	0.007070	0.983302	0.004480	0.004629
C	0.0000007	0.000006	0.000336	0.000227	0.001270	0.016185	0.955074	0.026899
Default	0	0	0	0	0	0	0	1